The Schrodinger equation for the interaction potential $x^2+\lambda x^2/(1+gx^2)$ and the first Heun confluent equation

1985 J. Phys. A: Math. Gen. 18 2441

(http://iopscience.iop.org/0305-4470/18/13/020)

J. Phys. A: Math. Gen. 18 (1985) 2441-2449. Printed in Great Britain

The Schrödinger equation for the interaction potential $x^2+\lambda x^2/(1+gx^2)$ and the first Heun confluent equation

G Marcilhacy† and R Pons‡

† Laboratoire de Physique Théorique, UA au CNRS no 769 Institut H Poincaré, 11 rue P et M Curie, 75231 Paris Cedex 05, France

‡ Physique Théorique de Plasmas (CNRS), Université P et M Curie, Tour 46-56, 5e étage, 4 Place Jussieu, 75230 Paris Cedex 05, France

Received 26 March 1984, in final form 2 January 1985

Abstract. We present systematically the solutions and the quasi-polynomial solutions for the Schrödinger equation with the $x^2+\lambda x^2/(1+gx^2)$ interaction potential with the help of the first Heun confluent equation or spheroidal Heun equation.

1. Introduction

In the present paper, we intend to investigate the solutions relative to singular points and polynomial and 'quasi-polynomial' solutions of the following one-dimensional Schrödinger equation

$$(\mathrm{d}^2/\mathrm{d}x^2+E-V(x))y(x)=0 \tag{1}$$

with the interaction potential,

$$V(x)\equiv x^2+\lambda x^2/(1+gx^2), \qquad -\infty<x<+\infty,$$

E is the eigenparameter and λ is a real number. With the intention of generalisation, contrary to the usual restraint [1-3], we do not put a restriction on the sign of the number g.

As is well known [1-5] this type of potential—when $g>0$—occurs in laser physics in the particular case of a toroidal plasma [7]. Moreover if $g=-g_1$, with $g_1>0$ and

However, there are cases for which the number g can take negative or imaginary values. This occurs when we take the differential equations in their canonical forms, namely the first Heun confluent equation or spheroidal Heun equation, e.g. in the description of a rotating black hole in general relativity [6] and for the electric field in the particular case of a toroidal plasma [7]. Moreover, if $g=-g_1$, with $g_1>0$ and $x=\mathrm{i}t$, (1) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2}-E-\left(t^2+\frac{\lambda t^2}{1+g_1t^2}\right)\right]y(t)=0. \tag{2}$$

This equation is formally the same as (1). As we shall see later the solutions of (2) which we study in this paper are, in spite of the above change of variables, all real.

Many authors [1-5] using various either numerical or analytic techniques have studied the solutions of (1). The purpose of this paper is to present systematically the

0305-4470/85/132441+09\$02.25

solutions of (1) in the vicinity of each singularity and the polynomial and quasi-polynomial solutions of (1), with the help of the confluent Heun function [8-10]. Up to now it does not seem that this method—to the best of our knowledge—has been used. It is possible to use this method to solve equations of the same type as (1), i.e. leading to any of the four forms of the confluent Heun function [9].

2. Heun's confluent equation. Canonical form of equation (1)

The Heun confluent equation has not been—to our knowledge—extensively studied in the literature. We do not know, for example, the integral representation and the analytic continuation of this Heun confluent function, but only integral equations for Heun confluent polynomials [8]. In both the case of the Heun equation and of the spheroidal wave equation [11], the main difficulty lies in the fact that the series representing the solutions yields a three-term recurrence relation which is not easy to solve. Nevertheless, we will give solutions which may be written as terminating polynomials for suitable values of the parameters, without studying the case where the parameters verifies a transcendent relation [11].

The Heun confluent equation with two regular singularities and an irregular singularity, at $+\infty$, may be written in the form,

$$y''+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\beta\right)y'+\frac{\alpha\beta z+p}{z(z-1)}y=0 \tag{3}$$

where α, β, γ, δ, p are constants.

A summary of the study and of the research of solutions of (3) may be found in [10]. If

$$y\equiv G(p,z)=\sum_{n=0}^{\infty} g_n z^n \tag{4}$$

we find immediately the recurrence relation for the coefficients of G (the Heun confluent function):

$$\begin{gathered} g_0=1, \qquad g_1=p/\gamma, \\ (n+2)(n+1+\gamma)g_{n+2}-[(n+1)(n+\gamma+\delta-\beta)+p]g_{n+1}-(n+\alpha)\beta g_n=0. \end{gathered} \tag{5}$$

According to the theory of the analytic functions, the series (4) is valid for $|z|<1$, $z=1$ being the nearest singular point, whatever the sign of g. Therefore it seems that the result expressed by Whitehead *et al* [3] on the validity of (4) is inexact.

It should be noted that the rational character of the coefficients of equation (3) is conserved by transformations of the type

$$y(z)=e^{\mu z}z^{\lambda}(z-1)^{\nu}u(z), \tag{6}$$

with the constants μ, λ, ν being suitably chosen in functions of the parameters α, β, γ, δ, p of equation (3) [9].

These transformations form a group with eight parameters. This important property of the Heun confluent equation should be noted in order to seek the whole polynomial and quasi-polynomial solutions of equation (3). We retain later on equation (3) as the canonical form of the Heun confluent equation.

3. General solution of equation (1) relative to the three singular points

We put $y(z) = e^{-x^2/2} W(z)$ in (1) where $z = -gx^2$. We obtain the canonical form of (1)

$$W'' + \left(\frac{1}{g} + \frac{1}{2z}\right) W' + \left(\frac{E-1}{4g} + \frac{(1-E+\lambda/g)}{4g} z\right) \frac{W}{z(z-1)} = 0 \tag{7}$$

this equation is of the form (3), with

$$\begin{aligned} &\beta = 1/g && \alpha = \tfrac{1}{4}(1-E+\lambda/g) \\ &\gamma = \tfrac{1}{2} && p = (E-1)/4g. \\ &\delta = 0. \end{aligned} \tag{8}$$

3.1. *General solution of equation (7) relative to the regular singular point $z = 0$*

$$W(z) = AG(p; \alpha, \beta, \gamma, \delta; z) + Bz^{1-\gamma} G(p_2; 1+\alpha-\gamma, \beta, 2-\gamma, \delta; z) \tag{9}$$

where A and B are two constants and

$$p_2 \equiv p + (1-\gamma)(\delta - \beta),$$

α, β, γ, δ, p are defined by (8). The solution (9) is valid for $|z| < 1$.

3.2. *General solution of equation (7) relative to the regular singular point, $z = 1$*

The substitution $\sigma = 1 - z$ reduces equation (7) to

$$W'' + \left(-\frac{1}{g} + \frac{1}{2(\sigma-1)}\right) W' + \left(\frac{\lambda}{4g^2} - \frac{1-E+\lambda/g}{4g}\sigma\right) \frac{W}{\sigma(\sigma-1)} = 0 \tag{10}$$

where $W' = dW/d\sigma$.

(a) The first particular solution of this equation, expressed with the help of the Heun confluent function, has the form

$$W_1(\sigma) = G(\lambda/4g^2; \tfrac{1}{4}(1-E+\lambda/g), -1/g, 0, \tfrac{1}{2}; \sigma) = \sum_{n=0}^{\infty} h_n \sigma^n,$$

with the corresponding recurrence relation,

$$h_0 = 0, \qquad h_1 = 1,$$

$$\begin{aligned} &(n+2)(n+1)h_{n+2} - [(n+1)(n+\tfrac{1}{2}+1/g) + \lambda/4g^2]h_{n+1} \\ &\qquad + (1/g)[n + \tfrac{1}{4}(1-E+\lambda/g)]h_n = 0. \end{aligned} \tag{11}$$

The series (11) converges for $|\sigma| < 1$, or $|z-1| < 1$.

(b) The research of the second solution is not as easy as in the first case. The exponent difference in $\sigma = 0$ being zero, this second solution involves generally a logarithmic term.

We put $W(\sigma) = \chi(\sigma) W_1(\sigma)$ in (10). It follows that

$$\chi'' W_1 + \chi' \left[2W_1' + \left(-\frac{1}{g} + \frac{1}{2(\sigma-1)}\right) W_1\right] = 0. \tag{12}$$

We find immediately the first integral,

$$\chi' = \frac{K}{W_1^2}\frac{e^{\sigma/g}}{(1-\sigma)^{1/2}} \tag{13}$$

where K is a constant. Taking account of (11), we put

$$W_1(\sigma) = \sigma y(\sigma), \tag{14}$$

where

$$y(\sigma) = 1 + h_2\sigma + \ldots .$$

Equation (13) becomes thus,

$$\chi' = \frac{K}{\sigma^2}\left(\frac{1}{y^2}\frac{e^{\sigma/g}}{(1-\sigma)^{1/2}}\right), \tag{15}$$

the function

$$g(\sigma) \equiv \frac{1}{y^2} e^{\sigma/g}/(1-\sigma)^{1/2}$$

is regular in $\sigma = 0$ and therefore,

$$g(\sigma) = c_0 + c_1\sigma + c_2\sigma^2 + \ldots$$

where

$$c_n = (1/n!)g^{(n)}(0).$$

Thus equation (14) becomes,

$$\chi' = K(c_0/\sigma^2 + c_1/\sigma + c_2 + c_3\sigma + \ldots)$$

and we find a solution χ of the form,

$$\chi = K(-c_0/\sigma + c_1 \log \sigma + c_2\sigma + \tfrac{1}{2}c_3\sigma^2 + \ldots + a) \tag{16}$$

where a is a constant,

$$c_0 = g(0) = 1 \qquad c_1 = g'(0) = -\lambda/4g^2.$$

In (16) we see that a logarithmic case arises, unless $c_1 = 0$, that is to say $\lambda = 0$. Finally we obtain the second solution of (10) in the form

$$W_2(\sigma) = W_1(\sigma) \log \sigma + \sum_{n=0}^{\infty} k_n\sigma^n \tag{17}$$

with

$$W_1(\sigma) = \sum_{n=1}^{\infty} h_n\sigma^n$$

and the recurrence relation for the coefficients h_i and k_j:

$$(n+2)(n+1)k_{n+2} - [(n+1)(n+\tfrac{1}{2}+1/g) + \lambda/4g^2]k_{n+1} + [(1/4g)(1+E-\lambda/g) + n/g]k_n$$
$$= -(2n+3)h_{n+2} + (2n+\tfrac{3}{2}+1/g)h_{n+1} - (1/g)h_n,$$

with $k_0 = 4g^2/\lambda$.

The constant a has been chosen in order to $k_1 = 0$. Taking account of (11) and (17), the general solution of (10) can be written

$$W = AW_1 + BW_2.$$

3.3. *General solution of equation (7) relative to the irregular singular point at infinity*

We assume solutions of (7) in the form,

$$W = z^s \sum_{n=0}^{\infty} d_n z^{-n} \qquad \text{with } d_0 \neq 0. \tag{18}$$

The point at infinity being an irregular singularity, we find only for s, the value $s = -\alpha$, where α is given by (8). The recurrence relation which the coefficients d_i must satisfy is now

$$\beta d_1 - d_0[\alpha(\alpha + 1 - \gamma - \delta + \beta) + p] = 0$$

$$\beta(n+2)d_{n+2} - d_{n+1}[(\alpha + n + 1)(\alpha + n + e - \gamma - \delta + \beta) + p]$$
$$+ d_n[(\alpha + n)(\alpha + n + 1 - \gamma)] = 0$$

with $d_0 = 1$. Returning to the x variable, and taking account of relations (8) we find a first solution of (1) in the form,

$$y_1(x) = \mathrm{e}^{-x^2/2}(-gx^2)^{-\alpha} \sum_{n=0}^{\infty} \frac{d_n}{(-g)^n x^{2n}}$$

we see that $\forall \alpha$, the series $|y_1|$ converges at infinity.

A second independent solution is given by

$$y_2(z) = \mathrm{e}^{-\beta z} G(p + \beta\gamma; -\alpha + \gamma + \delta, -\beta, \gamma, 0; z),$$

where G is the Heun confluent function of the form (18). We find,

$$y_2(z) = \mathrm{e}^{-\beta z} z^{-(-\alpha+\gamma+\delta)} \sum_{n=0}^{\infty} \frac{e_n}{z^n}$$

or

$$y_2(x) = \mathrm{e}^{x^2/2}(-gx^2)^{\alpha-\gamma}(1 - e_1/gx^2 + \ldots)$$

where

$$\alpha - \gamma = \tfrac{1}{4}(-1 - E + \lambda/g).$$

But this second solution $y_2(x)$ diverges at infinity.

4. Polynomial and quasi-polynomial solutions of equation (7) at $z = 0$

Using the invariance property of the canonical form (7) we obtain the following solutions in the vicinity of $z = 0$ ($x = 0$) and consequently the polynomials and quasi-polynomial solutions:

$$y_1(x) = \mathrm{e}^{-x^2/2} G\left(\frac{E-1}{4g}; \frac{1}{4}\left(1 - E + \frac{\lambda}{g}\right), \frac{1}{g}, \frac{1}{2}, 0; -gx^2\right)$$

$$y_2(x)=e^{-x^2/2}(1+gx^2)G\left(\frac{E+2g-1}{4g};\frac{1}{4}\left(5-E+\frac{\lambda}{g}\right),\frac{1}{g},\frac{1}{2},2;-gx^2\right)$$

$$y_3(x)=e^{-x^2/2}xG\left(\frac{E-3}{4g};\frac{1}{4}\left(3-E+\frac{\lambda}{g}\right),\frac{1}{g},\frac{3}{2},0;-gx^2\right)$$

$$y_4(x)=e^{-x^2/2}x(1+gx^2)G\left(\frac{E+6g-3}{4g};\frac{1}{4}\left(7-E+\frac{\lambda}{g}\right),\frac{1}{g},\frac{3}{2},2;-gx^2\right)$$

$$y_5(x)=e^{x^2/2}G\left(\frac{E+1}{4g};\frac{1}{4}\left[1-\left(-E+\frac{\lambda}{g}\right)\right],-\frac{1}{g},\frac{1}{2},0;-gx^2\right)$$

$$y_6(x)=e^{x^2/2}(1+gx^2)G\left(\frac{E+2g+1}{4g};\frac{1}{4}\left[5-\left(-E+\frac{\lambda}{g}\right)\right],-\frac{1}{g},\frac{1}{2},2;-gx^2\right)$$

$$y_7(x)=e^{x^2/2}xG\left(\frac{E+3}{4g};\frac{1}{4}\left[3-\left(-E+\frac{\lambda}{g}\right)\right],-\frac{1}{g},\frac{3}{2},0;-gx^2\right)$$

$$y_8(x)=e^{x^2/2}x(1+gx^2)G\left(\frac{E+6g+3}{4g};\frac{1}{4}\left[7-\left(-E+\frac{\lambda}{g}\right)\right],-\frac{1}{g},\frac{3}{2},2;-gx^2\right).$$

Between these eight solutions, the following functional relations holds,

$$y_1=y_2=y_5=y_6, \qquad y_3=y_4=y_7=y_8.$$

We note here the natural presence of factors $1+gx^2$, already noted by Heading [5], x or $x(1+gx^2)$ in some solutions. These factors do not introduce a simplification in the treatment of solutions. They change only the parameters of the corresponding Heun confluent function.

(1) The recurrence relation for y_1 is the following

$$2g^2(n+1)(2n+1)g_{n+1}-g[2n(2gn-g-2)+E-1]g_n-[4g(n-1)+g-Eg+\lambda]g_{n-1}=0$$

g_n can be written in the form

$$g_n=\frac{(-1)^n D_n}{g^n(2n)!}, \qquad n=0,1,2,\ldots,(g_0=1)$$

where D_n is the following tridiagonal determinant [3, 5]

$$D_n\equiv\begin{vmatrix} [E-1] & [2\times1\times1] & & 0 & \\ [Eg-\lambda-g] & [E-1+2g-4] & [2\times2\times3] & & \\ 0 & \begin{bmatrix}Eg-\lambda-g\\ -1\times4g\end{bmatrix} & \ddots & \ddots & \\ & & \ddots & \begin{bmatrix}E-1+(n-2)\\ \times(4gn-10g-4)\end{bmatrix} & [2(n-1)(2n-3)] \\ & & & \begin{bmatrix}Eg-\lambda-g\\ -(n-2)4g\end{bmatrix} & \begin{bmatrix}E-1+(n-1)\\ \times\{4g(n+1)-10g-4\}\end{bmatrix}\end{vmatrix}.$$

(2) The recurrence relation for y_3 is

$$2g^2(n+1)(2n+3)g_{n+1}-g[2n(2gn+g-2)+E-3]g_n$$
$$-[4g(n-1)+3g-Eg+\lambda]g_{n-1}=0$$

with

$$g_n = \frac{(-1)^n E_n}{g^n (2n)!} \qquad n = 0, 1, 2, \ldots, g_0 = 1$$

where E_n is the following tridiagonal determinant,

$$E_n \equiv \begin{vmatrix} [E-3] & [2\times 1\times 3] & & 0 & \\ [Eg-\lambda-3g] & [E-3+6g-4] & & [2\times 2\times 5] & \\ 0 & \begin{bmatrix} Eg-\lambda-3g \\ -1\times 4g \end{bmatrix} & \ddots & & \ddots \\ & & \ddots & \begin{bmatrix} E-3+(n-2) \\ \times(4gn-6g-4) \end{bmatrix} & [2(n-1)(2n-1)] \\ & & & \begin{bmatrix} Eg-\lambda-3g \\ -(n-2)4g \end{bmatrix} & \begin{bmatrix} E-3+(n-1) \\ \times\{4g(n+1)-6g-4\} \end{bmatrix} \end{vmatrix}.$$

The Heun confluent function $G(p; \alpha, \beta, \gamma, \delta; z) = \Sigma_{n=0}^{\infty} g_n z^n$ becomes a polynomial solution of $d^0 N$, if

$$\alpha = -N \qquad \text{with } N, \text{integer} \geq 0$$

$$g_{N+1} = 0$$

g_{N+1} being a polynomial of degree $N+1$.

These conditions lead us to consider two cases, whatever the sign of g,

(1) $-E + \lambda/g < 0$

(2) $-E + \lambda/g > 0$

and in each of these cases, two types of fundamental solutions,

$$-E + \lambda/g \equiv 1 \pmod 4 \qquad \text{for the } y_1 \text{ solutions,}$$

$$-E + \lambda/g \equiv 3 \pmod 4 \qquad \text{for the } y_3 \text{ solutions.}$$

The corresponding solutions, for $N = 0, 1, 2$, are listed in the table.

5. Reality and sign of the roots of $D_n(\lambda) = 0$

It is of physical interest to know if the roots of the polynomial form $D_n(\lambda) = 0$ are all real.

On account of the correspondence between (2) and (1) we see immediately that—whatever the sign of y—using a lemma given by Arscott [11], the roots of $D_n(\lambda) = 0$ are all real and different. On the other hand the result given by Whitehead [3] and Heading [5] showing that the whole non-zero roots of $D_n(\lambda) = 0$ are negative, no longer holds when $g < 0$. This fact can be verified directly for simple values of N.

6. Conclusion

In the present paper we have given a systematic preservation of the solutions of (1) relative to the singular points and the polynomial and quasi-polynomial solutions,

	$-E+\lambda/g<0$		$-E+\lambda/g>0$
Solutions of type	$y_1=G[\ldots]$ or $y_2=(1+gx^2)G[\ldots]$	Solutions of type	$y_5=e^{x^2/2}G[\ldots]$ or $y_6=e^{x^2/x}(1+gx^2)G[\ldots]$
$-E+\lambda/g\equiv 1 \pmod 4$			
$E=1$	$y=1$	$E=-1$	$y=e^{x^2/2}$
$E=5$	$y=1-2x^2$	$E=-5$	$y=e^{x^2/2}(1+2x^2)$
$E=1-2g$	$y=1+gx^2$	$E=-1-2g$	$y=e^{x^2/2}(1+gx^2)$
$E=9$	$y=1-4x^2+\frac{4}{3}x^4$	$E=-9$	$y=e^{x^2/2}(1+4x^2+\frac{4}{3}x^4)$
$E=-7g+3\pm(25g^2-12g+4)^{1/2}$	$y=(1+gx^2)[1-\frac{1}{2}(E-1+2g)x^2]$	$E=-7g-3\pm(25g^2+12g+4)^{1/2}$	$y=e^{x^2/2}(1+gx^2)[1-\frac{1}{2}(E+2g+1)x^2]$
Solutions of type	$y_3=(-gx^2)^{1/2}G[\ldots]$ or $y_4=(-gx^2)^{1/2}(1+gx^2)G$	Solutions of type	$y_7=(-gx^2)^{1/2}e^{x^2/2}G[\ldots]$ or $y_8=(-gx^2)^{1/2}e^{x^2/2}(1+gx^2)G[\ldots]$
$-E+\lambda/g\equiv 3 \pmod 4$			
$E=3$	$y=(-gx^2)^{1/2}$	$E=-3$	$y=(-gx^2)^{1/2}e^{x^2/2}$
$E=7$	$y=(-gx^2)^{1/2}(1-\frac{2}{3}x^2)$	$E=-7$	$y=(-gx^2)^{1/2}e^{x^2/2}(1+\frac{2}{3}x^2)$
$E=3-6g$	$y=(-gx^2)^{1/2}(1+gx^2)$	$E=-3-6g$	$y=(-gx)^{1/2}e^{x^2/2}(1+gx^2)$
$E=11$	$y=1-\frac{4}{3}x^2+\frac{4}{15}x^4$	$E=-11$	$y=(-gx^2)^{1/2}e^{x^2/2}(1+\frac{4}{3}x^2+\frac{4}{15}x^4)$
$E=-13g+5\pm(49g^2-4g+4)^{1/2}$	$y=(-gx^2)^{1/2}(1+gx^2)[1-\frac{1}{6}(E+6g-3)x^2]$	$E=-13g-5\pm(49g^2+4g+4)^{1/2}$	$y=(-gx^2)^{1/2}e^{x^2/2}(1+gx^2)$ $\times[1-\frac{1}{6}(E+6g+3)x^2]$

without restriction in the sign of g, knowing that the equation is, in fact, a Heun confluent equation.

Of course, these results are already partly known. This method can be used with good purpose for other linear differential equations of mathematical physics, seeing that, in fact, these equations are, Heun confluent, biconfluent, double-confluent, or three-confluen functions [9, 12].

References

[1] Mitra A K 1978 *J. Math. Phys.* **19** 2018–22
[2] Lai C S and He Lin 1982 *J. Phys. A: Math. Gen.* **15** 1495–502
[3] Whitehead R R, Watt A, Flessas G P and Nagarajan M A 1982 *J. Phys. A: Math. Gen.* **15** 1217–26
[4] Kaushal R S 1979 *J. Phys. A: Math. Gen.* **12** L253–8
[5] Heading J 1982 *J. Phys. A: Math. Gen.* **15** 2355–67
[6] Teukolski S A 1973 *Astrophys. J.* **185** 635
[7] Cotsaftis M and Sy W N C 1983 *Phys. Lett.* **93A** 4, 193
[8] Lambe C G and Ward D R 1934 *Quart. J. Math. (Oxford)* **5**
[9] Decarreau A, Maroni P and Robert A 1978 *Ann. Soc. Sci. Bruxelles* t 92 III, 151–89
[10] Blandin J, Pons R and Marcilhacy G 1983 *Lett. Nuovo Cimento* **38** 561–8
[11] Arscott F M 1964 *Periodic Differential Equations* (Oxford: Pergamon)
[12] Lemieux A and Bose A K 1969 *Ann. Inst. H. Poincaré* X, **3** 259–7